A Generalization of the Stability of Functional Equations

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Abstract: The additive functional equation is the best-known functional equation. The stability question is: At which time that the solutions of the functional inequality are near to that of the strict functional equation? In the paper, we generalize the stability of the approximately additive mapping in a restricted domain. It is provided that the mapping which meets the additive equation approximately in a restricted domain is stable in entire space.

Keywords: stability; approximately additive mapping; restricted domain

1. Introduction

In Ref. [1], Ulam put forward the fundamental problem with regard to the stability of homomorphisms. In 1941, the first significant result concerning this subject was gave by Hyers [2] which established the stability of the additive mapping. After that, functional equations stability is generalized and developed by an increasing number of mathematicians in various directions (see [3-14]). In particular, Rassias presented a drastic generalization of Hyers's theorem and provided a generalized solving method for Ulam's problem [3].

So far, the stability of functional equations defined on entire space was studied. The enquiry about the stability of the additive mapping in a restricted domain is natural. Exactly speaking, whether a strict additive mapping close to a mapping satisfying the additive inequality in a restricted domain.

Skof gave a result that was a good answer to the above question [4]. Hyers confirmed a generalized stability theorem of the additive mapping in a restricted domain and employed it to the asymptotic derivability which plays a significant role in nonlinear analysis [5].

In the note, we prove some theorems on stability of approximate additive mapping, which generalize Skof's theorem.

2. Main Results

Theorem 1. Let E_1 be a real normed vector space, E_2 be a real Banach space. Given numbers $\varepsilon > 0$, $m \ge 0$, $r \ge 2$, $r \in N$ and p with $0 < p < 1$. If $f: E_1 \to E_2$ satisfies

$$\left\| f(\sum_{n=1}^{r} x_n) - \sum_{n=1}^{r} f(x_n) \right\| \le \varepsilon \sum_{n=1}^{r} \|x_n\|^p \tag{1}$$

for all $x_i \in E_1$, $1 \le i \le r$ with $\sum_{i=1}^{r} \|x_i\|^p \ge m^p$, then there is a unique additive mapping $\psi: E_1 \to E_2$ such that

$$\| f(x) - \psi(x) \| \le \frac{\varepsilon}{1 - r^{p-1}} \|x\|^p \tag{2}$$

for all $x \in E_1$ with $\|x\| > \frac{m}{r^{1/p}}$. Moreover, ψ is presented by the formula

$$\psi(x) = \lim_{n\to\infty} \frac{1}{r^n} [f(r^n x)] \tag{3}$$

for all $x \in E_1$.

Proof. Claim that when $\|x\| > \frac{m}{r^{1/p}}$, we have

$$\left\| \frac{f(r^n x)}{r^n} - f(x) \right\| \le \varepsilon \|x\|^p \sum_{m=0}^{n-1} (r^{p-1})^m \tag{4}$$

for any integer n. We use induction on n. The case *n=1*, when $\|x\| > \frac{m}{r^{1/p}}$, that is when $r\|x\|^p \ge m^p$, we may put $x_1 = x_2 = \ldots = x_r = x$ in (1) to obtain

$$\left\| \frac{f(rx)}{r} - f(x) \right\| \le \varepsilon \|x\|^p .$$

Assume that (4) holds for some $n \in N$ and we want to prove it for the case $n+1$. Because in (4), we can replace x by rx since $\|rx\| > \frac{m}{r^{1/p}}$, we get

$$\left\| \frac{f(r^n rx)}{r^n} - f(rx) \right\| \le \varepsilon \|rx\|^p \sum_{m=0}^{n-1} (r^{p-1})^m .$$

Then

$$\left\| \frac{f(r^{n+1} x)}{r^{n+1}} - \frac{f(rx)}{r} \right\| \le \varepsilon \|x\|^p \sum_{m=1}^{n} (r^{p-1})^m .$$

We obtain that

$$\left\| \frac{f(r^{n+1} x)}{r^{n+1}} - f(x) \right\| \le \left\| \frac{f(r^{n+1} x)}{r^{n+1}} - \frac{f(rx)}{r} \right\| + \left\| \frac{f(rx)}{r} - f(x) \right\|$$

$$\le \varepsilon \|x\|^p \sum_{m=1}^{n} (r^{p-1})^m + \varepsilon \|x\|^p = \varepsilon \|x\|^p \sum_{m=0}^{n} (r^{p-1})^m .$$

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Thus for any integer n, we arrive at (4). Because $\sum_{m=0}^{\infty}(r^{p-1})^m$ converges to $\frac{1}{1-r^{p-1}}$ as $0<p<1$, we get

$$\left\|\frac{f(r^{n+1}x)}{r^{n+1}}-f(x)\right\|\le\frac{1}{1-r^{p-1}}\varepsilon\|x\|^p.$$

However for $m>n>0$,

$$\left\|\frac{f(r^m x)}{r^m}-\frac{f(r^n x)}{r^n}\right\|=\frac{1}{r^n}\left\|\frac{f(r^m x)}{r^{m-n}}-f(r^n x)\right\|$$

$\le\frac{1}{r^n}\frac{1}{1-r^{p-1}}\varepsilon\|r^n x\|^p$. So $\lim_{m,n\to\infty}\left\|\frac{f(r^m x)}{r^m}-\frac{f(r^n x)}{r^n}\right\|=0$.

Since E_2 is complete, thus the sequence $\{\frac{1}{r^n}[f(r^n x)]\}$ converges.

Let $g(x)=\lim_{n\to\infty}\frac{1}{r^n}[f(r^n x)]$, then when $\|x\|>\frac{m}{r^{1/p}}$,

$$\|g(x)-f(x)\|\le\frac{1}{1-r^{p-1}}\varepsilon\|x\|^p \tag{5}$$

Now we suppose that $\|x_i\|\ge\frac{m}{r^{1/p}}$ and $\left\|\sum_{i=1}^{r}x_i\right\|\ge\frac{m}{r^{1/p}}$, then for all $n\in N$, by (1) we have that

$$g(\sum_{k=1}^{r}x_k)=\sum_{k=1}^{r}g(x_k).\ \left\|\frac{1}{r^n}f(\sum_{i=1}^{r}r^n x_i)-\frac{1}{r^n}\sum_{i=1}^{r}f(r^n x_i)\right\|$$

$$\le\frac{\varepsilon}{r^{n(1-p)}}\sum_{i=1}^{r}\|x_i\|^p.$$

By the definition of g, we get that for all $x_i\in E_1, 1\le i\le r$ with $\|x_i\|\ge\frac{m}{r^{1/p}}$ and $\left\|\sum_{i=1}^{r}x_i\right\|\ge\frac{m}{r^{1/p}}$,

Given any $x\in E_1$ with $0<\|x\|\le\frac{m}{r^{1/p}}$, let $k=k(x)=\min\{k\in N: r^k\|x\|\ge\frac{m}{r^{1/p}}\}$. And set

$$\psi(x)=\begin{cases}0, .. & x=0;\\ r^{-k}g(r^k x),. & 0<\|x\|<\frac{m}{r^{1/p}};\\ g(x), & \|x\|\ge\frac{m}{r^{1/p}}..\end{cases}$$

then ψ is an extension of g to E_1.

Now we prove that for every $x\in E_1$

$$\psi(x)=\lim_{n\to\infty}\frac{1}{r^n}[f(r^n x)].$$

Fixed an x in E_1 with $0<\|x\|<\frac{m}{r^{1/p}}$, we can see

Case 1. If $0<\|rx\|<\frac{m}{r^{1/p}}$, then $k(rx)=k(x)-1=k-1$ and by definition we have

$\psi(rx)=r^{-(k-1)}g(r^{k-1}(rx))=r^{-k}rg(r^k x)=r\psi(x)$. Case 2. If $\|rx\|\ge\frac{m}{r^{1/p}}$, then $k(x)=1$ and $k(rx)=0$. So $\psi(rx)=g(rx)=r^{-1}rg(rx)=r\psi(x)$. Then we get that for all x in E_1, $\psi(rx)=r\psi(x)$.

Let a nonzero element $x\in E_1$. Select a positive integer M so big satisfying $\|r^M x\|\ge\frac{m}{r^{1/p}}$. Then

$$\psi(x)=r^{-M}g(r^M x)\ =r^{-M}\lim_{n\to\infty}\frac{1}{r^n}f(r^n\cdot r^M x)$$

$$=\lim_{n\to\infty}\frac{1}{r^{n+M}}f(r^{n+M}x)$$

which proves (3) for $x\ne 0$. Obviously (3) is true for $x=0$.

Nest we will prove the additivity of ψ. When either x or y is zero, we can easily come out that ψ is additive. Suppose that $x\ne 0$ and $y\ne 0$, choose integer $n>0$ with $\|r^n x\|^p+\|r^n y\|^p>m^p$, then we have from (1) that

$$\|f(r^n x+r^n y+0+\cdots+0)$$

$$-f(r^n x)-f(r^n y)-(r-2)f(0)\|\le\varepsilon(\|r^n x\|^p+\|r^n y\|^p).$$

Divide two sides of this inequality by r^n, we compute that $\psi(x+y)=\psi(x)+\psi(y)$.

So ψ is an additive mapping. Since when $\|x\|\ge\frac{m}{r^{1/p}}$, $\psi(x)=g(x)$, then by (5) we obtain (2).

The last we prove that ψ is unique. Assume there is an additive mapping $\phi: E_1\to E_2$ satisfies (2) when $\|x\|\ge\frac{m}{r^{1/p}}$, then for all $x\in E_1\setminus\{0\}$, we deduce that

$$\|\phi(nx)-\psi(nx)\|\le\frac{2\varepsilon}{1-r^{p-1}}\|x\|^p$$

for the big enough n. Therefore

$$\|\phi(x)-\psi(x)\|\le\frac{2\varepsilon}{n(1-r^{p-1})}\|x\|^p.$$

Let $n\to\infty$, we find that $\phi(x)=\psi(x),\ \forall x\in E_1\setminus\{0\}$. Obviously, $\phi(0)=\psi(0)$.

Corollary 1. Let E_1 be a real normed vector space, E_2 be a real Banach space. Given numbers $\varepsilon>0$, $m\ge 0$, $r\ge 2$, $r\in N$ and p with $0\le p<1$. If $f: E_1\to E_2$ satisfies

$$\|g(x)-f(x)\|\le\frac{1}{1-r^{p-1}}\varepsilon\|x\|^p \tag{5}$$

$$\left\|f(\sum_{n=1}^{r}x_n)-\sum_{n=1}^{r}f(x_n)\right\|\le\varepsilon\sum_{n=1}^{r}\|x_n\|^p \tag{6}$$

for all $x_i\in E_1$, $1\le i\le r$ with $\|x_i\|\ge m$ and $\sum_{i=1}^{r}\|x_i\|\ge m$. Then there is a unique additive function $\psi: E_1\to E_2$ such that

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$$\| f(x)-\psi(x)\| \le \frac{\varepsilon}{1-r^{p-1}}\|x\|^p \qquad (7)$$

for all $x \in E_1$ with $\|x\| > m$.

Proof. We replace $\|x\| > \frac{m}{r^{1/p}}$ with $\|x_i\| \ge m$ and the proof is similar with the proof of Theorem 1.

By using the result of Corollary 1 we can get the following result.

Corollary 2. Let E_1 be a real normed vector space, E_1 be a real Banach space and $\varepsilon \ge 0$, $m \ge 0$, $r \ge 2$, $r \in N$. If $f: E_1 \to E_2$ satisfies

$$\left\| f(\sum_{n=1}^{r} x_n) - \sum_{n=1}^{r} f(x_n) \right\| \le \varepsilon \qquad (8)$$

for all $x_i \in E_1$, $1 \le i \le r$ with $\sum_{i=1}^{r}\|x_i\| \ge m$. Then there is a unique additive mapping $\psi: E_1 \to E_2$ such that

$$\| f(x)-\psi(x)\| \le \frac{\varepsilon}{r-1} \qquad (9)$$

for all $x \in E_1$ with $\|x\| > \frac{m}{r}$, and

$$\| f(x)-\psi(x)\| \le 9\varepsilon + 9(r-2)\| f(0)\|, \ \ \forall x \in E_1. \qquad (10)$$

Proof. From Corollary 1, we know that there exists an additive mapping $\psi: E_1 \to E_2$ satisfying (9) when $x \in E_1$ and $\|x\| \ge \frac{m}{r}$. On the other hand, when $\|x\| + \|y\| \ge m$,

$$\| f(x+y) - f(x) - f(y)\| \le \varepsilon + (r-2)\| f(0)\|.$$

Therefore, by F. Skof's theorem in [4], there is an additive mapping $\varphi : E_1 \to E_2$ such that

$$\| f(x)-\varphi(x)\| \le 9\varepsilon + 9(r-2)\| f(0)\|, \forall x \in E_1. \qquad (11)$$

Thus, for each $x \in E_1 \setminus \{0\}$, when n is big enough, we have $\|\varphi(nx)-\psi(nx)\| \le 9\varepsilon + \frac{\varepsilon}{r-1} + 9(r-2)\| f(0)\|$.

Therefore

$$\|\varphi(x)-\psi(x)\| \le \frac{9(r-2)\| f(0)\|}{n} + \frac{9\varepsilon}{n} + \frac{\varepsilon}{n(r-1)}.$$

Let $n \to \infty$, we see that $\varphi(x) = \psi(x)$. Obviously, $\varphi(0) = \psi(0)$. Thus, from (11), we obtain (10).

3. Conclusion

This paper mainly discuss the stability of functional equations. We generalize the stability of the approximately additive mapping in a restricted domain. And we prove that the mapping which meets the additive equation approximately in a restricted domain is stable in entire space. These results generalize Skof's theorem.

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